

Seiberg–Witten prepotential for E-string theory and global symmetries

Kazuhiro Sakai

Yukawa Institute for Theoretical Physics, Kyoto University

Kyoto 606-8502, Japan

`ksakai@yukawa.kyoto-u.ac.jp`

Abstract

We obtain Nekrasov-type expressions for the Seiberg–Witten prepotential for the six-dimensional (1,0) supersymmetric E-string theory compactified on T^2 with non-trivial Wilson lines. We consider compactification with four general Wilson line parameters, which partially break the E_8 global symmetry. In particular, we investigate in detail the cases where the Lie algebra of the unbroken global symmetry is $E_n \oplus A_{8-n}$ with $n = 8, 7, 6, 5$ or D_8 . All our Nekrasov-type expressions can be viewed as special cases of the elliptic analogue of the Nekrasov partition function for the $SU(N)$ gauge theory with $N_f = 2N$ flavors. We also present a new expression for the Seiberg–Witten curve for the E-string theory with four Wilson line parameters, clarifying the connection between the E-string theory and the $SU(2)$ Seiberg–Witten theory with $N_f = 4$ flavors.

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1. Introduction

The E-string theory is one of the simplest interacting quantum field theories with $(1,0)$ supersymmetry in six dimensions [1–5]. It is obtained as the low energy theory of the heterotic string theory on K3 when an E_8 instanton shrinks to zero size [1, 2]. The theory is decoupled from gravity. It is probably a conventional local quantum field theory, though it does not have a Lagrangian description. Another unusual feature is that fundamental excitations are strings, called E-strings, rather than particles. The moduli space of vacua consists of a Coulomb branch with one tensor multiplet and a Higgs branch with 29 hypermultiplets. There are no vector multiplets and E_8 appears as a global symmetry group.

The E-string theory shows extremely rich properties when toroidally compactified down to lower dimensions [3–12]. In the compactified theories one can break the E_8 global symmetry by coupling its currents to the background E_8 gauge field with nontrivial Wilson lines. For each circle of the toroidal compactification there are eight Wilson line parameters taking their values in the Cartan torus of E_8 . By turning on these parameters, one can break E_8 to its subgroups and realize models with various global symmetries.

When the theory is toroidally compactified down to four dimensions, the low energy dynamics in the Coulomb branch is described by Seiberg–Witten theory [13, 14]. The Seiberg–Witten curve was constructed with the most general Wilson line parameters [4, 15]. Recently, it was found that the Seiberg–Witten prepotential admits a Nekrasov-type expression [16]. The expression is for the case with no Wilson line parameters. In this paper we present a Nekrasov-type expression with four general Wilson line parameters. It is verified up to a sufficiently high order (involving Young diagrams with 10 boxes) that the prepotential given by this expression is in perfect agreement with that computed from the Seiberg–Witten curve.

The Seiberg–Witten curve with full eight Wilson line parameters takes a rather complicated form. In the case of four Wilson line parameters, however, the curve reduces to a much simpler expression. Interestingly, it is expressed in terms of the curve for the $SU(2)$ Seiberg–Witten theory with $N_f = 4$ flavors. It has been known that the low energy theory of the E-string theory on T^2 can flow to that of the four-dimensional $SU(2)$ $N_f = 4$ theory. In particular, it was argued that the $SL(2, \mathbb{Z})$ duality of the latter theory is derived from the $SL(2, \mathbb{Z})$ action on the T^2 [4]. Our expression clarifies how it occurs, in particular how the $SO(8)$ triality emerges from the E-string theory.

By adjusting four Wilson line parameters to special values, one can realize the cases where the Lie algebra of the unbroken global symmetry is $E_n \oplus A_{8-n}$ with $n = 8, 7, 6, 5$ or D_8 . We present explicit forms of the Seiberg–Witten curves and the Nekrasov-type expressions for these specific cases. In each of these cases the Seiberg–Witten prepotential counts multiplicities of BPS E-strings wound around one of the circles of the toroidal compactification with general winding numbers and momenta. The multiplicities are equivalent to Gromov–Witten invariants associated with the E_n del Pezzo surface or $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in a Calabi–Yau threefold [3, 9, 17, 18]. Our Nekrasov-type expressions provide us with the generating functions of these invariants in very simple, closed forms. In particular, these expressions respect the modular properties of the partition functions of wound BPS E-strings [5, 11, 12].

As in the case with no Wilson line parameters [16], our Nekrasov-type expression can be viewed as a special case of the elliptic analogue of the Nekrasov partition function for the $SU(4)$ gauge theory with $N_f = 8$ flavors [19, 20]. Moreover, for particular values of the Wilson line parameters our expression can be embedded in the elliptic analogue of the Nekrasov partition functions for the $SU(N)$ $N_f = 2N$ theories with $N = 3, 2$. In fact, for all the cases with $E_n \oplus A_{8-n}$ with $n = 8, 7, 6, 5$ and D_8 the Nekrasov-type expressions can be viewed as special cases of the elliptic Nekrasov partition function for the $SU(3)$ $N_f = 6$ theory. Furthermore, expressions for the cases with $E_7 \oplus A_1$, $E_5 \oplus A_3$ and D_8 can also be viewed as special cases of the elliptic Nekrasov partition function for the $SU(2)$ $N_f = 4$ theory.

The organization of this paper is as follows. In section 2, we present our new expression for the Seiberg–Witten curve with four general Wilson line parameters and discuss its properties. In section 3, we present the Nekrasov-type expression with four Wilson line parameters. We then focus on some particular cases in which the general formula reduces to a sum over fewer partitions. In section 4, we investigate in detail the cases with global symmetries $E_n \oplus A_{8-n}$ with $n = 8, 7, 6, 5$ and D_8 . Conventions of special functions are summarized in Appendix A.

2. Seiberg–Witten curve with four Wilson line parameters

In this section we present a new expression for the Seiberg–Witten curve for the E-string theory compactified on T^2 with four Wilson line parameters. We clarify how it is related to the Seiberg–Witten curve for the four-dimensional $SU(2)$ gauge theory with $N_f = 4$ flavors.

The Seiberg–Witten curve for the E-string theory compactified on T^2 with the

most general Wilson line parameters was constructed in [4,15]. (An improved expression in terms of E_8 -invariant Jacobi forms is available in [21].) It takes the following form

$$y^2 = 4x^3 - fx - g \quad (2.1)$$

with

$$f = \sum_{j=0}^4 a_j u^{4-j}, \quad g = \sum_{j=0}^6 b_j u^{6-j}. \quad (2.2)$$

The coefficients a_j, b_j depend on nine complex parameters, τ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_8)$. τ is the complex modulus of the T^2 and $\boldsymbol{\mu}$ are the Wilson line parameters which specify the background E_8 gauge field along the T^2 [6]. In this paper we restrict ourselves to the cases in which Wilson line parameters take the form

$$\boldsymbol{\mu} = (m_1, m_2, m_3, m_4, m_1, m_2, m_3, m_4) \quad (2.3)$$

or

$$\boldsymbol{\mu} = (0, 0, 0, 0, m_1 + m_2, m_1 - m_2, m_3 + m_4, m_3 - m_4). \quad (2.4)$$

These two configurations are related to each other by a sequence of E_8 Weyl reflections and thus correspond to the same curve. Keeping this embedding in mind, we hereafter specify the Wilson line parameters by the following short notation

$$\boldsymbol{m} = (m_1, m_2, m_3, m_4). \quad (2.5)$$

The most general Seiberg–Witten curve for the E-string theory is invariant under the automorphism group consisting of affine E_8 Weyl group and $\text{SL}(2, \mathbb{Z})$ [22]. From this one can easily deduce the automorphism of the curve in the present setup as follows. The curve with four Wilson line parameters is invariant under the following transformations

$$\bullet \ m_i \leftrightarrow m_j \quad \text{for any } i \neq j, \quad (2.6)$$

$$\bullet \ m_i \leftrightarrow -m_i \quad \text{for any } i, \quad (2.7)$$

$$\bullet \ m_i \rightarrow m_i - \frac{1}{2} \sum_{j=1}^4 m_j \quad \text{for all } i = 1, \dots, 4, \quad (2.8)$$

$$\bullet \ \boldsymbol{m} \rightarrow \boldsymbol{m} + \boldsymbol{w}, \quad \boldsymbol{w} \in \Gamma_w. \quad (2.9)$$

Here Γ_w denotes the weight lattice of D_4 ,

$$\Gamma_w := \left\{ \boldsymbol{w} \in \mathbb{Z}^4 \cup \left(\mathbb{Z} + \frac{1}{2} \right)^4 \right\}. \quad (2.10)$$

The group generated by the above transformations is in fact the affine automorphism group of Γ_w . There is also an $\text{SL}(2, \mathbb{Z})$ automorphism generated by the following transformations

$$\bullet \tau \rightarrow \tau + 1, \quad (2.11)$$

$$\bullet \tau \rightarrow -\frac{1}{\tau}, \quad \mathbf{m} \rightarrow \frac{\mathbf{m}}{\tau} \quad \text{with} \quad (u, x, y) \rightarrow (\tau^{-6}Lu, \tau^{-10}L^2x, \tau^{-15}L^3y), \quad (2.12)$$

where

$$L := e^{2\pi i |\mathbf{m}|^2 / \tau}. \quad (2.13)$$

In principle, the explicit form of the curve with four Wilson line parameters is obtained by simply substituting (2.3) or (2.4) into the expression in [21]. However, the expression constructed in this way is rather complicated for practical purposes. In what follows we will express the same Seiberg–Witten curve in a more convenient form by means of the curve for the $\text{SU}(2)$ gauge theory with $N_f = 4$ fundamental hypermultiplets. Recall that the Seiberg–Witten curve for the $\text{SU}(2)$ $N_f = 4$ theory is given by [14]

$$\tilde{y}^2 = W_1 W_2 W_3 + A(W_1 T_1(e_2 - e_3) + W_2 T_2(e_3 - e_1) + W_3 T_3(e_1 - e_2)) - A^2 N \quad (2.14)$$

with

$$\begin{aligned} W_i &= \tilde{x} - e_i \tilde{u} - e_i^2 R, \\ A &= (e_1 - e_2)(e_2 - e_3)(e_3 - e_1), \\ R &= \frac{1}{2} \sum_i M_i^2, \\ T_1 &= \frac{1}{12} \sum_{i>j} M_i^2 M_j^2 - \frac{1}{24} \sum_i M_i^4, \\ T_2 &= -\frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2 M_j^2 + \frac{1}{48} \sum_i M_i^4, \\ T_3 &= \frac{1}{2} \prod_i M_i - \frac{1}{24} \sum_{i>j} M_i^2 M_j^2 + \frac{1}{48} \sum_i M_i^4, \\ N &= \frac{3}{16} \sum_{i>j>k} M_i^2 M_j^2 M_k^2 - \frac{1}{96} \sum_{i \neq j} M_i^2 M_j^4 + \frac{1}{96} \sum_i M_i^6, \\ e_1 &= \frac{\vartheta_3^4 + \vartheta_4^4}{12}, \quad e_2 = \frac{\vartheta_2^4 - \vartheta_4^4}{12}, \quad e_3 = \frac{-\vartheta_2^4 - \vartheta_3^4}{12}. \end{aligned} \quad (2.15)$$

Here $\vartheta_k := \vartheta_k(0, \tau)$ are the Jacobi theta functions (see Appendix A). τ denotes the complexified bare gauge coupling and M_1, \dots, M_4 are the masses of the fundamental

hypermultiplets. To obtain the Seiberg–Witten curve for the E-string theory, let us first make the following transformation of variables,

$$\begin{aligned}\tilde{u} &= -\frac{\eta^{24}}{l^2}(u + u_0) - \frac{E_6}{12E_4}R, \\ \tilde{x} &= -\frac{\eta^{24}}{l^2(u - u_0)}x + \frac{E_4}{72}R, \\ \tilde{y}^2 &= -\frac{\eta^{72}}{4l^6(u - u_0)^3}y^2.\end{aligned}\tag{2.16}$$

Here $E_{2k} := E_{2k}(\tau)$ and $\eta := \eta(\tau)$ are the Eisenstein functions and the Dedekind eta function respectively. The curve (2.14)–(2.15) can then be written in the form

$$y^2 = 4x^3 - (\tilde{a}_0 u^2 + \tilde{a}_1 u + \tilde{a}_2)(u - u_0)^2 x - (\tilde{b}_0 u^3 + \tilde{b}_1 u^2 + \tilde{b}_2 u + \tilde{b}_3)(u - u_0)^3.\tag{2.17}$$

Here \tilde{a}_j, \tilde{b}_j are some functions in τ , lM_i and u_0 . l is a parameter that gives an inverse mass scale. It can be absorbed in the definitions of $\tilde{u}, \tilde{x}, \tilde{y}$ and M_i , but let us keep it for later use. Next, we identify the parameters as

$$u_0 = \frac{1}{2\eta^{12}E_4} \sum_{\sigma \in S_4} \prod_{j=1}^4 \vartheta_j(m_{\sigma(j)}, \tau)^2\tag{2.18}$$

and

$$\begin{aligned}lM_1 &= \prod_{j=1}^4 \vartheta_1(m_j, \tau) - \prod_{j=1}^4 \vartheta_2(m_j, \tau), \\ lM_2 &= \prod_{j=1}^4 \vartheta_1(m_j, \tau) + \prod_{j=1}^4 \vartheta_2(m_j, \tau), \\ lM_3 &= \prod_{j=1}^4 \vartheta_3(m_j, \tau) - \prod_{j=1}^4 \vartheta_4(m_j, \tau), \\ lM_4 &= \prod_{j=1}^4 \vartheta_3(m_j, \tau) + \prod_{j=1}^4 \vartheta_4(m_j, \tau).\end{aligned}\tag{2.19}$$

In (2.18), σ denotes a permutation of $\{1, 2, 3, 4\}$ and the sum is taken over all such permutations. Under this identification the curve (2.17) coincides precisely with the Seiberg–Witten curve for E-string theory [15, 21] with the Wilson line parameters given by (2.3) or (2.4).

By reversing the above construction, one can reproduce the Seiberg–Witten curve for the $SU(2)$ $N_f = 4$ theory from that of the E-string theory. The reader might

think that the transformation (2.16) is artificial because x and y are rescaled by u -dependent factors. One could use the following linear transformation

$$\begin{aligned} u &= -\frac{l^2}{\eta^{24}} \left(\tilde{u} + \frac{E_6}{12E_4} R \right) - u_0, \\ x &= \frac{2l^2 u_0}{\eta^{24}} \left(\tilde{x} - \frac{E_4}{72} R \right), \\ y^2 &= \frac{32l^6 u_0^3}{\eta^{72}} \tilde{y}^2, \end{aligned} \tag{2.20}$$

instead of (2.16). The curve (2.14) is then obtained by taking the limit $l \rightarrow 0$. In fact, under this limit the identification (2.16) coincides with (2.20).

In the above reduction the bare gauge coupling τ of the $SU(2)$ $N_f = 4$ theory is identified with the complex modulus τ of T^2 on which the E-string theory is compactified. This means that the $SL(2, \mathbb{Z})$ duality of the $N_f = 4$ theory is identified with the $SL(2, \mathbb{Z})$ action of the T^2 [4]. Recall that the $SL(2, \mathbb{Z})$ duality of the $SU(2)$ $N_f = 4$ theory is mixed with $SO(8)$ triality [14]. That is, the spectrum of the theory is not invariant under the transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$, but is invariant under the combinations

$$\begin{aligned} \bullet \quad \tau \rightarrow \tau + 1 \quad \text{with} \quad & \begin{aligned} M_1 &\rightarrow M_1, \\ M_2 &\rightarrow M_2, \\ M_3 &\rightarrow M_3, \\ M_4 &\rightarrow -M_4 \end{aligned} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \bullet \quad \tau \rightarrow -\frac{1}{\tau} \quad \text{with} \quad & \begin{aligned} M_1 &\rightarrow \frac{1}{2}(M_1 + M_2 + M_3 - M_4), \\ M_2 &\rightarrow \frac{1}{2}(M_1 + M_2 - M_3 + M_4), \\ M_3 &\rightarrow \frac{1}{2}(M_1 - M_2 + M_3 + M_4), \\ M_4 &\rightarrow \frac{1}{2}(-M_1 + M_2 + M_3 + M_4). \end{aligned} \end{aligned} \tag{2.22}$$

In (2.19) M_i are identified with functions in τ and m_j . Modular transformations of these functions precisely reproduce the above transformations of M_i (up to an overall factor which can be absorbed into l). This peculiar identification was first found in [22], where a different connection between the two theories was considered.

The identification (2.19) admits the following interpretation in connection with the automorphism group of the curve. We saw that the automorphism group of the

present Seiberg–Witten curve is governed by the weight lattice of D_4 denoted by $\Gamma_{\mathbf{w}}$. This lattice can be viewed as the union of four sublattices

$$\Gamma_{\mathbf{w}} = \Gamma_{\mathbf{b}} \cup \Gamma_{\mathbf{v}} \cup \Gamma_{\mathbf{s}} \cup \Gamma_{\mathbf{c}}, \quad (2.23)$$

where

$$\begin{aligned} \Gamma_{\mathbf{b}} &:= \{\mathbf{w} = (w_1, w_2, w_3, w_4) \in \mathbb{Z}^4 \mid \sum_{j=1}^4 w_j \in 2\mathbb{Z}\}, \\ \Gamma_{\mathbf{v}} &:= \{\mathbf{w} = (1, 0, 0, 0) + \mathbf{v} \mid \mathbf{v} \in \Gamma_{\mathbf{b}}\}, \\ \Gamma_{\mathbf{s}} &:= \{\mathbf{w} = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \mathbf{v} \mid \mathbf{v} \in \Gamma_{\mathbf{b}}\}, \\ \Gamma_{\mathbf{c}} &:= \{\mathbf{w} = (-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \mathbf{v} \mid \mathbf{v} \in \Gamma_{\mathbf{b}}\}. \end{aligned} \quad (2.24)$$

In terms of these sublattices, (2.19) can be expressed as

$$\begin{aligned} lM_1 &= -2\Theta_{\mathbf{c}}(\tau, \mathbf{m}), & lM_2 &= 2\Theta_{\mathbf{s}}(\tau, \mathbf{m}), \\ lM_3 &= 2\Theta_{\mathbf{v}}(\tau, \mathbf{m}), & lM_4 &= 2\Theta_{\mathbf{b}}(\tau, \mathbf{m}), \end{aligned} \quad (2.25)$$

where $\Theta_{\mathcal{R}}(\tau, \mathbf{m})$ is the theta function for sublattice $\Gamma_{\mathcal{R}}$,

$$\Theta_{\mathcal{R}}(\tau, \mathbf{m}) := \sum_{\mathbf{w} \in \Gamma_{\mathcal{R}}} \exp(\pi i \mathbf{w}^2 \tau + 2\pi i \mathbf{w} \cdot \mathbf{m}). \quad (2.26)$$

Note that $\Theta_{\mathcal{R}}(\tau, \mathbf{m})/\eta(\tau)^4$ with $\mathcal{R} = \mathbf{b}, \mathbf{v}, \mathbf{s}, \mathbf{c}$ respectively give the characters of the basic, vector, spinor, conjugate-spinor representations of the affine D_4 algebra. Therefore (2.25) means that the masses of the hypermultiplets in $SU(2)$ $N_f = 4$ theory are essentially identified with these affine D_4 characters.

Note that the D_4 symmetry acting on the Wilson line parameters m_j should not be confused with the D_4 symmetry acting on the masses M_i . The two D_4 symmetries are related in a nontrivial manner. For instance, the exchange of M_2 for M_3 is an element of the Weyl group of the latter D_4 . We see from (2.25) that this corresponds to the exchange of $\Gamma_{\mathbf{s}}$ for $\Gamma_{\mathbf{v}}$, which is an outer automorphism of the former D_4 .

In the rest of this section let us sketch out how to calculate the prepotential from the Seiberg–Witten curve. Our Seiberg–Witten curve given by (2.14)–(2.19) is expressed in the Weierstrass form. An elliptic curve in the Weierstrass form can be parametrized as

$$y^2 = 4x^3 - \frac{1}{12} \frac{E_4(\tilde{\tau})}{\omega^4} x - \frac{1}{216} \frac{E_6(\tilde{\tau})}{\omega^6}. \quad (2.27)$$

Here $\tilde{\tau}$ is the complex structure modulus and ω (multiplied by 2π) is one of the fundamental periods of the elliptic curve. By comparing this expression with the

explicit form of the Seiberg–Witten curve, one can calculate $\omega(u, \tau, \mathbf{m}), \tilde{\tau}(u, \tau, \mathbf{m})$ as series expansions in $1/u$. They are related to the scalar vev φ and the prepotential F_0 by

$$\partial_u \varphi = \frac{i}{2\pi} \omega, \quad (2.28)$$

$$\partial_\varphi^2 F_0 = 8\pi^3 i (\tilde{\tau} - \tau). \quad (2.29)$$

These relations parametrically determine the function $F_0(\varphi, \tau, \mathbf{m})$. The integration constants are determined accordingly. The reader is referred for the details of these calculations to [21].

3. Nekrasov-type expression with four Wilson line parameters

In this section we present an explicit expression for the Seiberg–Witten prepotential for the E-string theory with four Wilson lines and discuss its properties.

3.1. General expression

Let $\mathbf{R}^{(N)} = (R_1, \dots, R_N)$ denote an N -tuple of partitions. Each partition R_k is a nonincreasing sequence of nonnegative integers

$$R_k = \{\nu_{k,1} \geq \nu_{k,2} \geq \dots \geq \nu_{k,\ell(R_k)} > \nu_{k,\ell(R_k)+1} = \nu_{k,\ell(R_k)+2} = \dots = 0\}. \quad (3.1)$$

Here the number of nonzero $\nu_{k,i}$ is denoted by $\ell(R_k)$. R_k is represented by a Young diagram. We let $|R_k|$ denote the size of R_k , i.e. the number of boxes in the Young diagram of R_k :

$$|R_k| := \sum_{i=1}^{\infty} \nu_{k,i} = \sum_{i=1}^{\ell(R_k)} \nu_{k,i}. \quad (3.2)$$

Similarly, the size of $\mathbf{R}^{(N)}$ is denoted by

$$|\mathbf{R}^{(N)}| := \sum_{k=1}^N |R_k|. \quad (3.3)$$

We let $R_k^\vee = \{\nu_{k,1}^\vee \geq \nu_{k,2}^\vee \geq \dots\}$ denote the conjugate partition of R_k . We also introduce the notation

$$h_{k,l}(i, j) := \nu_{k,i} + \nu_{l,j}^\vee - i - j + 1, \quad (3.4)$$

which represents the relative hook-length of a box at (i, j) between the Young diagrams of R_k and R_l .

In our expression we consider a sum over four partitions. For our present purpose, it is convenient to express these partitions as

$$\mathbf{R}^{(4)} = (R_1, R_2, R_3, R_4) = (R_{11}, R_{10}, R_{00}, R_{01}). \quad (3.5)$$

The prepotential is then given by

$$F_0 = (2\hbar^2 \ln \mathcal{Z})|_{\hbar=0}, \quad (3.6)$$

where

$$\mathcal{Z} = \sum_{\mathbf{R}^{(4)}} Q^{|\mathbf{R}^{(4)}|} \prod_{a,b,c,d} \prod_{(i,j) \in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar + m_{cd}, \tau\right) \vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar - m_{cd}, \tau\right)}{\vartheta_{1-|a-c|, 1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau\right)^2} \quad (3.7)$$

and

$$Q := e^{2\pi i \varphi + \pi i \tau}. \quad (3.8)$$

Here the sum is taken over all possible partitions $\mathbf{R}^{(4)}$ (including the empty partition). Indices a, b, c, d take values 0, 1, while a set of indices (i, j) run over the coordinates of all boxes in the Young diagram of R_{ab} . $\vartheta_{ab}(z, \tau)$ are the Jacobi theta functions (see Appendix A). $h_{ab,cd}(i, j)$ is the relative hook-length defined between partitions R_{ab} and R_{cd} . m_{ab} are the Wilson line parameters, which are identified with those appearing in the Seiberg–Witten curve by

$$\mathbf{m} = (m_1, m_2, m_3, m_4) = (m_{11}, m_{10}, m_{00}, m_{01}). \quad (3.9)$$

If we set $\mathbf{m} = \mathbf{0}$, the expression reduces to the one studied in [16].

We find that the above F_0 coincides with the prepotential computed from the Seiberg–Witten curve in the last section. We verified it by computing the series expansion of F_0 in Q independently by each of the methods and comparing the coefficients up to order Q^{10} . In doing this, the following identities

$$\begin{aligned} & \vartheta_{ab}(m+z, \tau) \vartheta_{ab}(m-z, \tau) \\ &= \vartheta_{00}(2m, 2\tau) \vartheta_{a0}(2z, 2\tau) + (-1)^b \vartheta_{10}(2m, 2\tau) \vartheta_{1-a,0}(2z, 2\tau) \end{aligned} \quad (3.10)$$

$$\begin{aligned} &= \frac{1}{2} \left[\left(\frac{\vartheta_{00}(2m, 2\tau)}{\vartheta_{a0}(0, 2\tau)} + (-1)^b \frac{\vartheta_{10}(2m, 2\tau)}{\vartheta_{1-a,0}(0, 2\tau)} \right) \vartheta_{00}(z, \tau)^2 \right. \\ & \quad \left. + (-1)^a \left(\frac{\vartheta_{00}(2m, 2\tau)}{\vartheta_{a0}(0, 2\tau)} - (-1)^b \frac{\vartheta_{10}(2m, 2\tau)}{\vartheta_{1-a,0}(0, 2\tau)} \right) \vartheta_{01}(z, \tau)^2 \right] \end{aligned} \quad (3.11)$$

turn out to be useful. Using these identities one can rewrite both the Seiberg–Witten curve and the Nekrasov-type expression in such a way that all the dependence on

m_{cd} is expressed through $\vartheta_{00}(2m_{cd}, 2\tau)$ and $\vartheta_{10}(2m_{cd}, 2\tau)$. The comparison can then be made in the same way as in the case of $\mathbf{m} = \mathbf{0}$ by using the Taylor expansions of the theta functions [16].

As in [16], one can express \mathcal{Z} as a special case of the elliptic analogue of the Nekrasov partition function for the $SU(N)$ gauge theory with $N_f = 2N$ fundamental hypermultiplets [19, 20]

$$\begin{aligned} & \mathcal{Z}_{N_f=2N}^{SU(N)}(\hbar; \varphi, \tau; a_1, \dots, a_N; m_1, \dots, m_{2N}) \\ &:= \sum_{\mathbf{R}^{(N)}} (-e^{2\pi i \varphi})^{|\mathbf{R}^{(N)}|} \prod_{k=1}^N \prod_{(i,j) \in R_k} \frac{\prod_{n=1}^{2N} \vartheta_1(a_k + m_n + \frac{1}{2\pi}(j-i)\hbar, \tau)}{\prod_{l=1}^N \vartheta_1(a_k - a_l + \frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau)^2}. \end{aligned} \quad (3.12)$$

In terms of this function, (3.7) can be expressed as

$$\mathcal{Z} = \mathcal{Z}_{N_f=8}^{SU(4)}\left(\hbar; \varphi, \tau; 0, \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; m_1, m_2, m_3, m_4, -m_1, -m_2, -m_3, -m_4\right). \quad (3.13)$$

In the following sections we will see that this type of notation provides us with an efficient, universal way of expressing various Nekrasov-type formulas for specific cases.

Currently we do not have a good physical explanation why the instanton counting of $SU(4)$ $N_f = 8$ type accounts for the BPS spectrum of the E-string theory. From the technical point of view the elliptic analogue of the Nekrasov partition function with four colors and eight flavors is perfect for reproducing the expansion $F_0 = \sum_{n=1}^{\infty} Z_n Q^n$ with $Z_1 = \frac{1}{2}\eta^{-12} \sum_{k=1}^4 \prod_{i=1}^8 \vartheta_k(\mu_i, \tau)$ [5]. No other known elliptic Nekrasov partition functions [20] seem to have an immediate connection with the above form of Z_1 . For particular values of Wilson line parameters, however, one can express \mathcal{Z} in terms of the elliptic analogues of the $SU(N)$ $N_f = 2N$ Nekrasov partition functions with $N = 3, 2$, as we will see in the next subsection. We have not examined whether the BPS counting of the E-string theory has any connection with the instanton counting of other types of gauge groups, for which no explicit elliptic Nekrasov partition functions are known.

The prepotential F_0 for the E-string theory represents the genus zero topological string amplitude for a family of local $\frac{1}{2}\text{K3}$ [5]. However, as was observed in the case of $\mathbf{m} = \mathbf{0}$ [16], higher order parts of the expansion $\ln \mathcal{Z} = \frac{1}{2}F_0\hbar^{-2} + \dots$ do not give higher genus amplitudes [12, 21, 23]. The disagreement can be clearly seen as the difference of modular anomalies. With the help of (3.11) one immediately sees that \mathcal{Z} exhibits the same modular anomaly as in the case of $\mathbf{m} = \mathbf{0}$. This deviates from

the modular anomaly of the all-genus topological string partition function for the local $\frac{1}{2}\text{K3}$ starting at genus one.

3.2. Reductions to sums over fewer partitions

For particular values of the Wilson line parameters the above Nekrasov-type sum over partitions reduces to that over fewer partitions.

Let us first consider the case where one of the four Wilson line parameters is set to be zero,

$$\mathbf{m} = (0, m_{10}, m_{00}, m_{01}). \quad (3.14)$$

In this case, the product in the sum in (3.7) vanishes for any $\mathbf{R}^{(4)}$ with $R_{11} \neq \{0\}$. This is because the Young diagram of $R_{11} \neq \{0\}$ always contains a box at $(i, j) = (1, 1)$, where the theta functions in the numerator become $\vartheta_{11}(0, \tau) = 0$ for $(c, d) = (1, 1)$. Hence, \mathcal{Z} is actually a sum over three partitions

$$\mathbf{R}^{(3)} = (R_{10}, R_{00}, R_{01}). \quad (3.15)$$

This structure has already been found in the case of no Wilson line parameters [16]. Furthermore, recall that for any function $f(x)$ the following identity holds:

$$\prod_{(i,j) \in R_k} f(h_{k,l}(i, j)) = \prod_{(i,j) \in R_k} f(j - i) \quad \text{if } R_l = \{0\}. \quad (3.16)$$

This identity can be easily shown by regarding the product over $j = 1, \dots, \nu_{k,i}$ as that over $\tilde{j} := \nu_{k,i} - j + 1 = 1, \dots, \nu_{k,i}$ on the left-hand side. Due to this identity, one sees that the expression for \mathcal{Z} reduces to the form

$$\mathcal{Z} = \sum_{\mathbf{R}^{(3)}} Q^{|\mathbf{R}^{(3)}|} \prod_{(a,b),(c,d)} \prod_{(i,j) \in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar + m_{cd}, \tau\right) \vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar - m_{cd}, \tau\right)}{\vartheta_{1-|a-c|, 1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i, j)\hbar, \tau\right)^2}. \quad (3.17)$$

This is almost identical to (3.7), except that the sum is now over $\mathbf{R}^{(3)}$ and indices $(a, b), (c, d)$ take values $(1, 0), (0, 0), (0, 1)$ only. In terms of the elliptic Nekrasov partition function (3.12), the above simplification is expressed as

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}_{N_f=8}^{\text{SU}(4)} \left(\hbar; \varphi, \tau; 0, \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, m_{10}, m_{00}, m_{01}, 0, -m_{10}, -m_{00}, -m_{01} \right) \\ &= \mathcal{Z}_{N_f=6}^{\text{SU}(3)} \left(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; m_{10}, m_{00}, m_{01}, -m_{10}, -m_{00}, -m_{01} \right). \end{aligned} \quad (3.18)$$

As we will see in the next section, this simplified \mathcal{Z} encompasses all the cases of global symmetries $E_n \oplus A_{8-n}$ with $n = 8, 7, 6, 5$ and D_8 .

Next, let us further restrict ourselves to the cases with

$$\mathbf{m} = \left(0, \frac{1}{2}, m_1, m_2\right). \quad (3.19)$$

In this setting, the expression for \mathcal{Z} reduces to the form

$$\mathcal{Z} = \sum_{\mathbf{R}^{(2)}} Q^{|\mathbf{R}^{(2)}|} \prod_{k,l=1}^2 \prod_{(i,j) \in R_k} \frac{\vartheta_{k+2}\left(\frac{1}{2\pi}(j-i)\hbar + m_l, \tau\right) \vartheta_{k+2}\left(\frac{1}{2\pi}(j-i)\hbar - m_l, \tau\right)}{\vartheta_{|k-l|+1}\left(\frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2}, \quad (3.20)$$

where $\mathbf{R}^{(2)} = (R_1, R_2)$. In terms of the elliptic Nekrasov partition function (3.12), \mathcal{Z} with (3.19) can be expressed as

$$\begin{aligned} \mathcal{Z} &= \mathcal{Z}_{N_f=8}^{\text{SU}(4)} \left(\hbar; \varphi, \tau; 0, \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, \frac{1}{2}, m_1, m_2, 0, -\frac{1}{2}, -m_1, -m_2 \right) \\ &= \mathcal{Z}_{N_f=6}^{\text{SU}(3)} \left(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; \frac{1}{2}, m_1, m_2, -\frac{1}{2}, -m_1, -m_2 \right) \\ &= \mathcal{Z}_{N_f=4}^{\text{SU}(2)} \left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; m_1, m_2, -m_1, -m_2 \right). \end{aligned} \quad (3.21)$$

As we will see in the next section, the cases of global symmetries $E_7 \oplus A_1$, $E_5 \oplus A_3$ and D_8 are realized as special cases of this setting.

Furthermore, if we set

$$\mathbf{m} = \left(0, \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}\right), \quad (3.22)$$

\mathcal{Z} vanishes. This is consistent with the fact that the corresponding unbroken global symmetry is $D_4 \oplus D_4$ and the Seiberg–Witten curve in this case describes a constant elliptic fibration over the moduli space [22].

4. Two-parameter families

In this section we consider the cases in which the Lie algebra of the unbroken global symmetry is $E_{9-N} \oplus A_{N-1}$ with $N = 1, 2, 3, 4$ or D_8 . These cases are of particular interest because the prepotential in each case generates Gromov–Witten invariants associated with the E_{9-N} del Pezzo surface or $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in a Calabi–Yau threefold. In particular, we consider two-parameter families of Calabi–Yau, whose prepotentials depend not only on Kähler modulus φ but also on another Kähler modulus τ . These two-parameter families have been studied by means of mirror symmetry [3, 12].

All of the above global symmetries are maximal regular subalgebras of E_8 and one can easily find the corresponding values of Wilson line parameters [22]. As we saw in section 2, different values of Wilson line parameters \mathbf{m} related by the transformations (2.6)–(2.9) correspond to the same Seiberg–Witten curve. We will present a representative of \mathbf{m} for each of these cases.

Substituting each of these \mathbf{m} into our general expression given by (2.14)–(2.19) one obtains the Seiberg–Witten curve for each of the cases. The result can be simplified by making use of theta function identities. We will present the final form of the curve after making a shift of variables x and u . The shift leads to a further simplification. The curve without the shift can easily be recovered by first eliminating the quadratic term in x by a shift of x and then eliminating the cubic part in u of the linear term in x by a shift of u . The Seiberg–Witten curve for the E-string theory describes an elliptic fibration over \mathbb{P}^1 with singular fibers. Using the Weierstrass form of the curve one can easily check that the types of singular fibers correspond precisely to the simple Lie algebras constituting the unbroken global symmetry [22].

We will also present explicit Nekrasov-type expressions for each of the cases. The prepotential is obtained from \mathcal{Z} through (3.6). Following [5] we introduce the winding number expansion of the prepotential by

$$F_0(\varphi, \tau) = \sum_{n=1}^{\infty} Q^n Z_n(\tau). \quad (4.1)$$

Z_n for $E_{9-N} \oplus A_{N-1}$ with $N = 1, 2, 3, 4$ can be expressed in terms of $E_2(\tau)$ and modular forms of $\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid a \equiv d \equiv 1, c \equiv 0 \pmod{N} \right\}$. We will introduce generators α_N, β_N of modular forms of $\Gamma_1(N)$ as well as a function λ_N and present explicit forms of Z_n for small n . In the D_8 case, Z_n are expressed in terms of $E_2(\tau)$ and modular forms of $\Gamma_1(2)$.

In each of the above cases, the prepotential can be expressed as

$$F_0(\varphi, \tau) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{1}{m^3} e^{2\pi i m(n\varphi + k\tau)}. \quad (4.2)$$

Integer $N_{n,k}$ represents the multiplicity of BPS E-strings wound around one of the circles of the toroidal compactification with winding number n and momentum k . Up to an overall normalization the values of $N_{n,n}$ turn out to be equal to the genus zero Gromov–Witten invariants associated with the E_{9-N} del Pezzo surface or $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in a Calabi–Yau threefold [3, 9, 17, 18]. Our formulas also generate invariants $N_{n,k}$ with $k \neq n$, where k is the degree associated with the homology class of the

elliptic fiber. It would be very interesting to see how our combinatorial expressions are related to the geometric computation of these invariants [18].

4.1. E_8

Let us first consider the case with an E_8 global symmetry. This is the case originally discussed in [16] and is realized by the trivial Wilson line parameters

$$\mathbf{m} = (0, 0, 0, 0). \quad (4.3)$$

The corresponding Seiberg–Witten curve is given by

$$y^2 = 4x^3 - \frac{1}{12}E_4u^4x - \frac{1}{216}E_6u^6 + 4u^5. \quad (4.4)$$

The Nekrasov-type expression can be written as

$$\mathcal{Z} = \mathcal{Z}_{N_f=6}^{\text{SU}(3)} \left(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, 0, 0, 0, 0, 0 \right) \quad (4.5)$$

$$= \sum_{\mathbf{R}^{(3)}} Q^{|\mathbf{R}^{(3)}|} \prod_{(a,b),(c,d)} \prod_{(i,j) \in R_{ab}} \frac{\vartheta_{ab} \left(\frac{1}{2\pi}(j-i)\hbar, \tau \right)^2}{\vartheta_{1-|a-c|, 1-|b-d|} \left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau \right)^2}. \quad (4.6)$$

Here the set of indices $(a, b), (c, d)$ take values $(1, 0), (0, 0), (0, 1)$ and we let the three partitions be denoted by $\mathbf{R}^{(3)} = (R_{10}, R_{00}, R_{01})$. The first three coefficients of the expansion (4.1) are

$$\begin{aligned} Z_1 &= \lambda_1 \alpha_1, \\ Z_2 &= \lambda_1^2 \alpha_1 \left(\frac{\alpha_1 E_2 + 2\beta_1}{24} \right), \\ Z_3 &= \lambda_1^3 \alpha_1 \left(\frac{54\alpha_1^2 E_2^2 + 216\alpha_1 \beta_1 E_2 + 109\alpha_1^3 + 197\beta_1^2}{15552} \right), \end{aligned} \quad (4.7)$$

where

$$\alpha_1 := E_4, \quad \beta_1 := E_6 \quad (4.8)$$

and

$$\lambda_1 := \frac{1}{\eta^{12}}. \quad (4.9)$$

These Z_n agree with the original results [11]. Table 1 shows the values of $N_{n,k}$ for low n and k . These numbers were originally computed by using mirror symmetry [3].

	k	0	1	2	3	4	5	\dots
n								
1		1	252	5130	54760	419895	2587788	
2		0	0	-9252	-673760	-20534040	-389320128	
3		0	0	0	848628	115243155	6499779552	
4		0	0	0	0	-114265008	-23064530112	
5		0	0	0	0	0	18958064400	
\vdots								\ddots

Table 1: BPS multiplicities $N_{n,k}$ for the E_8 case.

4.2. $E_7 \oplus A_1$

The $E_7 \oplus A_1$ symmetry is realized by the following Wilson line parameters

$$\mathbf{m} = \left(0, 0, 0, \frac{1}{2}\right). \quad (4.10)$$

The Seiberg–Witten curve is given by

$$y^2 = 4x^3 + (\vartheta_3^4 + \vartheta_4^4) u^2 x^2 + \left(\frac{\vartheta_3^4 \vartheta_4^4}{4} u - \frac{16}{\vartheta_3^2 \vartheta_4^2}\right) u^3 x. \quad (4.11)$$

The Nekrasov-type expression in this case takes a remarkably simple form

$$\mathcal{Z} = \mathcal{Z}_{N_f=4}^{\text{SU}(2)} \left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, 0, 0, 0 \right) \quad (4.12)$$

$$= \sum_{\mathbf{R}^{(2)}} Q^{|\mathbf{R}^{(2)}|} \prod_{k,l=1}^2 \prod_{(i,j) \in R_k} \frac{\vartheta_{k+2} \left(\frac{1}{2\pi} (j-i) \hbar, \tau \right)^2}{\vartheta_{|k-l|+1} \left(\frac{1}{2\pi} \hbar_{kl} (i,j) \hbar, \tau \right)^2}. \quad (4.13)$$

The first three coefficients of the expansion (4.1) are

$$\begin{aligned} Z_1 &= \lambda_2 \alpha_2, \\ Z_2 &= \lambda_2^2 \alpha_2 \left(\frac{\alpha_2 E_2 - \alpha_2^2 + 3\beta_2}{24} \right), \\ Z_3 &= \lambda_2^3 \alpha_2 \left(\frac{6\alpha_2^2 E_2^2 - 12\alpha_2^3 E_2 + 36\alpha_2 \beta_2 E_2 + 16\alpha_2^4 - 33\alpha_2^2 \beta_2 + 51\beta_2^2}{1728} \right), \end{aligned} \quad (4.14)$$

where

$$\alpha_2 := \frac{1}{2} (\vartheta_3^4 + \vartheta_4^4), \quad \beta_2 := \vartheta_3^4 \vartheta_4^4 = \frac{\eta(\tau)^{16}}{\eta(2\tau)^8} \quad (4.15)$$

and

$$\lambda_2 := \frac{\vartheta_3^2 \vartheta_4^2}{\eta^{12}} = \frac{1}{\eta(\tau)^4 \eta(2\tau)^4}. \quad (4.16)$$

As expected, these Z_n are in agreement with the results obtained in [12]. Table 2 shows the values of $N_{n,k}$ for low n and k . The values of $N_{n,k}$ multiplied by two agree with the rational instanton numbers of the E_7 model in [12].

k	0	1	2	3	4	5	\dots
n							
1	1	28	138	680	2359	7980	
2	0	0	-136	-2272	-23208	-167872	
3	0	0	0	1620	50067	824544	
4	0	0	0	0	-29216	-1316544	
5	0	0	0	0	0	651920	
\vdots							\ddots

Table 2: BPS multiplicities $N_{n,k}$ for the $E_7 \oplus A_1$ case.

4.3. $E_6 \oplus A_2$

The $E_6 \oplus A_2$ symmetry is realized by the following Wilson line parameters

$$\mathbf{m} = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \quad (4.17)$$

The Seiberg–Witten curve is given by

$$y^2 = 4x^3 + 3\alpha_3^2 u^2 x^2 + \frac{2}{3}\alpha_3 \left(\beta_3 u - \frac{27}{\beta_3}\right) u^3 x + \frac{1}{27} \left(\beta_3 u - \frac{27}{\beta_3}\right)^2 u^4, \quad (4.18)$$

where

$$\alpha_3 := \vartheta_3(0, 2\tau)\vartheta_3(0, 6\tau) + \vartheta_2(0, 2\tau)\vartheta_2(0, 6\tau), \quad \beta_3 := \frac{\eta(\tau)^9}{\eta(3\tau)^3}. \quad (4.19)$$

The Nekrasov-type expression is given by

$$\mathcal{Z} = \mathcal{Z}_{N_f=6}^{\text{SU}(3)} \left(\hbar; \varphi, \tau; \frac{1}{2}, -\frac{1+\tau}{2}, \frac{\tau}{2}; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right) \quad (4.20)$$

$$= \sum_{\mathbf{R}^{(3)}} Q^{|\mathbf{R}^{(3)}|} \prod_{(a,b),(c,d)} \prod_{(i,j) \in R_{ab}} \frac{\vartheta_{ab} \left(\frac{1}{2\pi}(j-i)\hbar + \frac{1}{3}, \tau \right) \vartheta_{ab} \left(\frac{1}{2\pi}(j-i)\hbar - \frac{1}{3}, \tau \right)}{\vartheta_{1-|a-c|, 1-|b-d|} \left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau \right)^2}, \quad (4.21)$$

where $(a,b), (c,d) = (1,0), (0,0), (0,1)$ and $\mathbf{R}^{(3)} = (R_{10}, R_{00}, R_{01})$. The first three coefficients of the expansion (4.1) are

$$\begin{aligned} Z_1 &= \lambda_3 \alpha_3, \\ Z_2 &= \lambda_3^2 \alpha_3 \left(\frac{\alpha_3 E_2 + 2\beta_3}{24} \right), \\ Z_3 &= \lambda_3^3 \alpha_3 \left(\frac{27\alpha_3^2 E_2^2 + 108\alpha_3 \beta_3 E_2 + 45\alpha_3^6 - 4\alpha_3^3 \beta_3 + 112\beta_3^2}{7776} \right), \end{aligned} \quad (4.22)$$

where

$$\lambda_3 := \frac{\beta_3}{\eta^{12}} = \frac{1}{\eta(\tau)^3 \eta(3\tau)^3}. \quad (4.23)$$

	k	0	1	2	3	4	5	\dots
n								
1		1	9	27	85	234	567	
2		0	0	-18	-164	-1026	-4968	
3		0	0	0	81	1377	13365	
4		0	0	0	0	-576	-14040	
5		0	0	0	0	0	5085	
\vdots								\ddots

Table 3: BPS multiplicities $N_{n,k}$ for the $E_6 \oplus A_2$ case.

Table 3 shows the values of $N_{n,k}$ for low n and k . The values of $N_{n,k}$ multiplied by three agree with the rational instanton numbers of the E_6 model in [12].

4.4. $E_5 \oplus A_3$

The $E_5 \oplus A_3$ symmetry is realized by the following Wilson line parameters

$$\mathbf{m} = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right). \quad (4.24)$$

The Seiberg–Witten curve and the Nekrasov-type expression in this case are given respectively by

$$y^2 = 4x^3 + \left((\vartheta_3^4 + \vartheta_4^4) u + \frac{64}{(\vartheta_3^2 + \vartheta_4^2)\vartheta_3^3\vartheta_4^3} \right) ux^2 + \left(\frac{\vartheta_3^2\vartheta_4^2}{2}u - \frac{16}{(\vartheta_3^2 + \vartheta_4^2)\vartheta_3^3\vartheta_4^3} \right)^2 u^2x \quad (4.25)$$

and

$$\mathcal{Z} = \mathcal{Z}_{N_f=4}^{\text{SU}(2)} \left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; \frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4} \right) \quad (4.26)$$

$$= \sum_{\mathbf{R}^{(2)}} Q^{|\mathbf{R}^{(2)}|} \prod_{k,l=1}^2 \prod_{(i,j) \in R_k} \frac{\vartheta_{k+2} \left(\frac{1}{2\pi}(j-i)\hbar + \frac{1}{4}, \tau \right) \vartheta_{k+2} \left(\frac{1}{2\pi}(j-i)\hbar - \frac{1}{4}, \tau \right)}{\vartheta_{|k-l|+1} \left(\frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau \right)^2}. \quad (4.27)$$

The first three coefficients of the expansion (4.1) are

$$\begin{aligned} Z_1 &= \lambda_4, \\ Z_2 &= \lambda_4^2 \left(\frac{E_2 + \alpha_4 + \beta_4}{24} \right), \\ Z_3 &= \lambda_4^3 \left(\frac{3E_2^2 + 6\alpha_4 E_2 + 6\beta_4 E_2 + 8\alpha_4^2 + 4\alpha_4\beta_4 + 5\beta_4^2}{864} \right), \end{aligned} \quad (4.28)$$

where

$$\alpha_4 := \vartheta_3(0, 2\tau)^4 = \frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}, \quad \beta_4 := \vartheta_4(0, 2\tau)^4 = \frac{\eta(\tau)^8}{\eta(2\tau)^4} \quad (4.29)$$

	k	0	1	2	3	4	5	...
n								
1		1	4	10	24	55	116	
2		0	0	-5	-32	-152	-576	
3		0	0	0	12	147	1056	
4		0	0	0	0	-48	-832	
5		0	0	0	0	0	240	
\vdots								\ddots

Table 4: BPS multiplicities $N_{n,k}$ for the $E_5 \oplus A_3$ case.

and

$$\lambda_4 := \frac{\vartheta_3(0, 2\tau)^2 \vartheta_4(0, 2\tau)^6}{\eta^{12}} = \frac{(\vartheta_3^2 + \vartheta_4^2) \vartheta_3^3 \vartheta_4^3}{2\eta^{12}} = \frac{\eta(2\tau)^4}{\eta(\tau)^4 \eta(4\tau)^4}. \quad (4.30)$$

Table 4 shows the values of $N_{n,k}$ for low n and k . The values of $N_{n,k}$ multiplied by four agree with the rational instanton numbers of the E_5 model in [12].

4.5. D_8

The D_8 symmetry is realized by the following Wilson line parameters

$$\mathbf{m} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right). \quad (4.31)$$

The Seiberg–Witten curve and the Nekrasov-type expression in this case are given respectively by

$$y^2 = 4x^3 + \left((\vartheta_3^4 + \vartheta_4^4)u + \frac{64}{\vartheta_3^4 \vartheta_4^4}\right)ux^2 + \frac{\vartheta_3^4 \vartheta_4^4}{4}u^4x \quad (4.32)$$

and

$$\mathcal{Z} = \mathcal{Z}_{N_f=4}^{\text{SU}(2)} \left(\hbar; \varphi, \tau; -\frac{1+\tau}{2}, \frac{\tau}{2}; 0, 0, \frac{1}{2}, -\frac{1}{2} \right) \quad (4.33)$$

$$= \sum_{\mathbf{R}^{(2)}} Q^{|\mathbf{R}^{(2)}|} \prod_{k,l=1}^2 \prod_{(i,j) \in R_k} \frac{\vartheta_3\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right) \vartheta_4\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)}{\vartheta_{|k-l|+1}\left(\frac{1}{2\pi}h_{kl}(i,j)\hbar, \tau\right)^2}. \quad (4.34)$$

The first three coefficients of the expansion (4.1) are

$$\begin{aligned} Z_1 &= \tilde{\lambda}_2, \\ Z_2 &= \tilde{\lambda}_2^2 \left(\frac{E_2 + 2\alpha_2}{24} \right), \\ Z_3 &= \tilde{\lambda}_2^3 \left(\frac{6E_2^2 + 24\alpha_2 E_2 + 25\alpha_2^2 + 9\beta_2}{1728} \right), \end{aligned} \quad (4.35)$$

	k	0	1	2	3	4	5	\dots
n								
1		1	-4	10	-24	55	-116	
2		0	0	-4	32	-152	576	
3		0	0	0	-12	147	-1056	
4		0	0	0	0	-48	832	
5		0	0	0	0	0	-240	
\vdots								\ddots

Table 5: BPS multiplicities $N_{n,k}$ for the D_8 case.

where α_2, β_2 are defined in (4.15) and

$$\tilde{\lambda}_2 := \frac{\vartheta_3^4 \vartheta_4^4}{\eta^{12}} = \frac{\eta(\tau)^4}{\eta(2\tau)^8} = \frac{2^4}{\vartheta_2^4}. \quad (4.36)$$

Table 5 shows the values of $N_{n,k}$ for low n and k . We observe that the values of $N_{n,n}$ are related to the genus zero Gromov–Witten invariants of the local $\mathbb{P}^1 \times \mathbb{P}^1$ as

$$N_{n,n} = \sum_{n_1+n_2=n} N_{n_1,n_2}^{\mathbb{P}^1 \times \mathbb{P}^1}. \quad (4.37)$$

Note that the values of $N_{n,k}$ for the D_8 case are very similar to those for the $E_5 \oplus A_3$ case. This has been explained by the similarity between the Picard–Fuchs operators for $\mathbb{P}^1 \times \mathbb{P}^1$ (which is equal to the quadric surface in \mathbb{P}^3) and those for the E_5 del Pezzo surface [9]. From the point of view of the E-string theory this may be explained by the similarity between the Weyl orbits of D_8 and those of $E_5 \oplus A_3 \cong D_5 \oplus D_3$.

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A. Conventions of special functions

The Jacobi theta functions are defined as

$$\vartheta_{ab}(z, \tau) := \sum_{n \in \mathbb{Z}} \exp \left[\pi i \left(n + \frac{a}{2} \right)^2 \tau + 2\pi i \left(n + \frac{a}{2} \right) \left(z + \frac{b}{2} \right) \right], \quad (\text{A.1})$$

where a, b take values 0, 1. We also use the notation

$$\begin{aligned} \vartheta_1(z, \tau) &:= -\vartheta_{11}(z, \tau), & \vartheta_2(z, \tau) &:= \vartheta_{10}(z, \tau), \\ \vartheta_3(z, \tau) &:= \vartheta_{00}(z, \tau), & \vartheta_4(z, \tau) &:= \vartheta_{01}(z, \tau). \end{aligned} \quad (\text{A.2})$$

The Dedekind eta function is defined as

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (\text{A.3})$$

where $q := e^{2\pi i \tau}$. The Eisenstein series are given by

$$E_{2n}(\tau) = 1 + \frac{2}{\zeta(1-2n)} \sum_{k=1}^{\infty} \frac{k^{2n-1} q^k}{1 - q^k}. \quad (\text{A.4})$$

We often abbreviate $\vartheta_k(0, \tau)$, $\eta(\tau)$, $E_{2n}(\tau)$ as ϑ_k , η , E_{2n} respectively.

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